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A Matrix Method for Estimating the Liapunov Exponent of One-Dimensional Systems

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Let τ: [0, 1] → [0, 1] be a piecewise monotonic expanding map. Then τ admits an absolutely continuous invariant measure μ. A result of Kosyakin and Sandler shows that μ can be approximated by a sequence of absolutely continuous measures μ_n invariant under piecewise linear Markov maps τ_n. Each τ_n is constructed on the inverse images of the turning points of τ. The easily computable measures μ_n are used to estimate the Liapunov exponent of τ. The idea of using Markov maps for estimating the Liapunov exponent is applied to both expanding and nonexpanding maps.

KEY WORDS: Liapunov exponent; piecewise monotonic map; Markov map; absolutely continuous univariant measure; negative Schawarzian.

1. INTRODUCTION

A chaotic system is one in which long-term prediction of the system state is impossible because inaccuracies in speicifying the initial state of the system are rapidly amplied in time. The exponential divergence of nearby trajectories is an important indicator of deterministic chaos.[1,4,16,17,18] The Liapunov exponent[16] describes the rate of increase of the perturbations of the initial conditions. Let $I = [0, 1]$. In this paper we consider one-dimensional systems defined by a map $\tau: I \to I$. In Ref. 16 it is shown that the rate of orbital divergence, called the Liapunov exponent, is given by

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log_2 |\tau'(\tau^i(x))|$$

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where x is the initial point and τ^i is the ith iterate of τ. If τ admits an absolutely continuous invariant measure μ with probability density function f, the Birkhoff ergodic theorem applied to the function $\phi(x) = \log_2 |\tau'(x)|$ yields

$$\lambda = \int_0^1 f(x) \log_2 |\tau'(x)| \, dx \tag{1}$$

We quote from Ref. 4: "The Lyapunov exponent is most easily understood in this form [Eq. (1)]: local stretching, determined by the logarithm of the magnitude of the slope, is weighted by the probability of encountering that amount of stretching."

Given the functional form of a one-dimensional map, Eq. (1) provides a means of estimating λ. Such estimations are done by using long orbits to approximate the density. In Refs. 1 and 3 such an approach is discussed for experimental data for a chemical reaction.[1,3] But there are limitations to this approach, and the method is usually not robust.[1]

One of the problems in applying Eq. (1) in experimental situations is the underlying assumption that an absolutely invariant measure exists. In Ref. 15 it is shown that even a simple unimodal map on I can possess an uncountable number of continuous (but not absolutely continuous) ergodic measures. Each one of these measures is "exhibited" by a dense orbit and indeed possesses a dense set of generic points (Ref. 20, Proposition 5.8). Hence, an orbit that produces a histogram is no indication that that histogram corresponds to an absolutely continuous invariant measure. In fact, it could be the histogram of any of the uncountably many other invariant measures.

Recently, time series approaches to estimating Liapunov exponents have been proposed.[1,2,4,5,10] For a continuous, dissipative, n-dimensional dynamical system, the ith Liapunov exponent is defined in terms of the growth rate of the ith principal axis $p_i(t)$ of an n-sphere of initial conditions, i.e.,

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}$$

In one dimension, this reduces to

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log_2 \frac{L(t)}{L(0)}$$

where $L(0)$ is an interval of initial conditions and $L(t)$ is the length of this interval at time t. If τ is not continuous, this method does not work, since ellipsoids are not necessarily transformed into ellipsoids.

In Section 2 we consider a class of piecewise monotonic, expanding maps, which are not necessarily continuous. These maps admit absolutely continuous invariant measures[9,21] and furthermore have the property that the absolutely continuous invariant measures can be approximated by the absolutely continuous invariant measures of piecewise linear Markov maps.[9] The measures of the piecewise linear Markov maps are obtained by finding the left eigenvectors of matrices[22] and hence can be used to estimate the Liapunov exponent.

In Section 4 we study nonexpanding maps and prove that under certain conditions the alogorithm of Section 3 is applicable to such maps.

2. NOTATION AND BACKGROUND MATERIAL

Let $\tau: I \to I$ be a nonsingular, measurable transformation and let $\mathbf{B}$ denote the Lebesgue measurable subsets of I. A measure μ defined on $(I, \mathbf{B})$ is absolutely continuous if there exists a function $f: I \to [0, \infty)$, which is integrable with respect to Lebesgue measure m, i.e., $f \in \mathbf{L}_1 \equiv \mathbf{L}_1(I, \mathbf{B}, m)$ and for which

$$\mu(S) = \int_S f(x)\, m(dx) \qquad \forall S \in \mathbf{B}$$

The measure μ is invariant (under τ) if $\mu(\tau^{-1}S) = \mu(S)$ for all $S \in \mathbf{B}$.

The Frobenius–Perron operator $P_\tau: \mathbf{L}_1 \to \mathbf{L}_1$ is defined by

$$(P_\tau f)(x) = \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(s)\, m(ds)$$

P_τ has proven to be a useful tool in the study of absolutely continuous invariant measures[21]. Its importance lies in the fact that each of its fixed points is a density of a measure invariant under τ, i.e., if $P_\tau f^* = f^*$, then

$$\mu(A) = \int_A f^*(x)\, m(dx)$$

is invariant under τ.

Definition 1. The map $\tau: I \to I$ is called Markov if there exist points $0 = c_0 < c_1 < \cdots < c_{n-1} < c_n = 1$ such that for $i = 0, 1, \dots, n-1$, $\tau|_{I_i}$, where $I_i = (c_{i-1}, c_i)$, is a homeomorphism onto some interval $(c_{j(i)}, c_{k(i)})$. The partition of I defined by $\{I_i\}_{i=1}^n$ is referred to as a Markov partition (with respect to τ).

Let τ be a piecewise linear Markov map. Then P_τ, when restricted to the space of step functions on the Markov partition, is a matrix M_τ[22], with entries

$$m_{ij} = \begin{cases} 1/|\tau'|_{I_i}| & \text{if } \tau(I_i) \supset I_j \\ 0 & \text{otherwise} \end{cases}$$

In Ref. 27 it is shown that M_τ is similar to a stochastic matrix and therefore has a fixed point f which is a step function on the Markov partition.

Definition 2. Let $\{\mu_n\}$ be a sequence of absolutely continuous invariant probability measures and let f_n be the density of μ_n. We say $f_n(\mu_n)$ converges weakly to the density f (measure μ) if and only if for each $g \in C$, the space of real continuous functions on I,

$$\int_I g(x)\, f_n(x)\, m(dx) \to \int_I g(x)\, f(x)\, m(dx)$$

as $n \to \infty$. [It is sufficient that g is in a space that is dense in C (Ref. 12, Theorem 12.2).]

3. PIECEWISE MONOTONIC EXPANDING MAPS

Let the map $\tau: I \to I$ satisfy the following two conditions.

(a) There exist points $0 = a_0 < a_1 < \cdots < a_{N-1} < a_N = 1$ such that $\tau(x)$ is twice continuously differentiable on the intervals (a_i, a_{i+1}), $i = 0, 1, \ldots, N-1$, and at the points $\{a_i\}$ there are one-sided first derivatives.

(b) There are constants K, d, and M such that

$$K \geqslant |\tau'(x)| \geqslant d > 1$$

and

$$|\tau''(x)| \leqslant M$$

everywhere except at the points $\{a_i\}$, $i = 0, 1, \ldots, N$.

We let $\mathbf{C}$ denote this class of maps. Note that $\tau \in \mathbf{C}$ does not have to be continuous.

Let $\mathbf{I} = \{I_0, I_1, \ldots, I_{N-1}\}$ be the subintervals of the partition of I defined by $\{a_i\}$. Let

$$\mathbf{I}^{(k)} \equiv \bigvee_{i=0}^{k} \tau^{-i}(\mathbf{I})$$

denote the subintervals of the partition obtained by using all the points $\bigcup_{i=0}^{k} \tau^{-i}\{a_0,..., a_N\}$, and let $Q^{(k)} = \{q_1^{(k)},..., q_{r_k}^{(k)}\}$ be the set of end points of the intervals that are elements of $\mathbf{I}^{(k)}$. For an arbitrary point a_j, $j = 0, 1,..., N$, we consider the points

$$q_{i_{a_j^+}}^{(k)} \quad \text{and} \quad q_{i_{a_j^-}}^{(k)}$$

of the set $Q^{(k)}$, approximating a_j from the left and from the right, respectively. For a_0, it is

$$q_{i_{a_0^+}}^{(k)}$$

while for a_N it is

$$q_{i_{a_N^-}}^{(k)}$$

Given $\tau: I \to I$, we want to construct a Markov map $\bar{\tau}_k$ on $Q^{(k)}$ that approximates τ. To do this, we proceed as follows: if $q_i^{(k)} \neq a_j$ for some j, define $\bar{\tau}_k(q_i^{(k)}) = \tau(q_i^{(k)})$. It remains only to define $\bar{\tau}_k$ on the original partition points $\{a_0, a_1,..., a_N\}$. Consider $q_i^{(k)} = a_j$ for some j. If τ is increasing on (a_{j-1}, a_j), define $\bar{\tau}_k(a_j^-)$ to be that point $q_{i_{b_j}}^{(k)}$ of $Q^{(k)}$ that is closest to $\tau(a_j^-)$ and greater than or equal to $\tau(a_j^-)$. (See Fig. 1a.) Similarly, if τ is decreasing on (a_{j-1}, a_j), define $\bar{\tau}_k(a_j^-)$ to be the point $q_{i_{b_j}}^{(k)}$ of $Q^{(k)}$ that is closest to $\tau(a_j^-)$ and less than or equal to $\tau(a_j^-)$. (See Fig. 1b.) Also, we define $\bar{\tau}_k(a_j^+)$ to be that point $q_{i_{c_j}}^{(k)}$ of $Q^{(k)}$ that is nearest to $\tau(a_j^+)$ and less than or equal to $\tau(a_j^+)$ if τ is increasing on (a_j, a_{j+1}). If τ is decreasing on (a_j, a_{j+1}), define $\bar{\tau}_k(a_j^+)$ to be the point $q_{i_{c_j}}^{(k)}$ of $Q^{(k)}$ that is closest to $\tau(a_j^+)$ and greater than or equal to $\tau(a_j^+)$.

We now define $\bar{\tau}_k$ by the following conditions: $\bar{\tau}_k = \tau$ everywhere except on the intervals

$$(q_{i_{a_j^-}}^{(k)}, a_j) \qquad \text{and} \qquad (a_j, q_{i_{a_j^+}}^{(k)})$$

on which τ is changed in such a way that $\bar{\tau}_k \in \mathbf{C}$,

$$\lim_{x \to a_j^-} \bar{\tau}_k(x) = q_{i_{b_j}}^{(k)}, \qquad \lim_{x \to a_j^+} \bar{\tau}_k(x) = q_{i_{c_j}}^{(k)}$$

(This choice of $\bar{\tau}_k$ makes it expanding and guarantees that the approximating sequence of absolutely continuous invariant measures is weakly compact.) From the proof of Lemma 3 of Ref. 9, we obtain that $\bar{\tau}_k$ can be replaced by a piecewise linear Markov map τ_k having the same endpoints as $\bar{\tau}_k$.

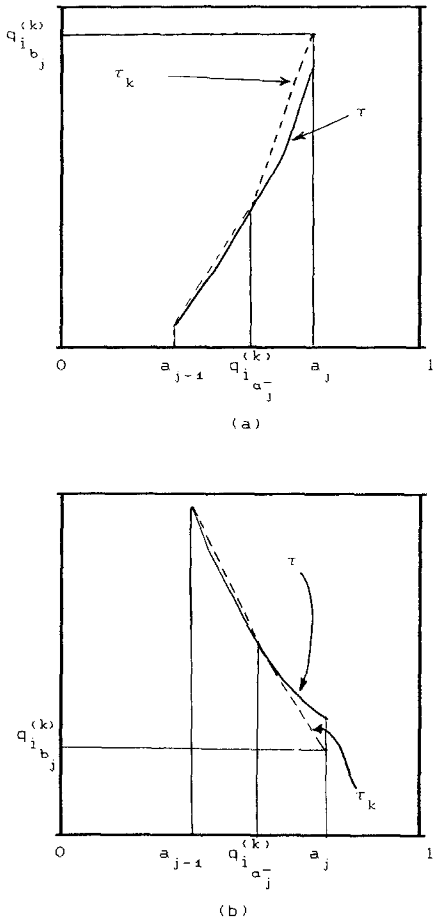

Fig. 1. Piecewise linear Markov map approximation for an expanding map.

The following result is proved in Ref. 9.

Theorem 1. Let $\tau \in \mathbf{C}$ and μ be an absolutely continuous measure invariant under τ with probability density function f. Then $\tau_k \to \tau$ uniformly, each τ_k admits an absolutely continuous invariant measure with density f_k, and $f_k \to f$ weakly as $k \to \infty$.

Note that τ is erogodic with respect to μ.

Corollary 1. Let

$$\lambda_k = \int_0^1 f_k(x) \log_2 |\tau_k'(x)| \, dx$$

Then

$$\lim_{k \to \infty} \lambda_k = \lambda \equiv \int_0^1 f(x) \log_2 |\tau'(x)| \, dx$$

Proof. Since $\tau \in \mathbf{C}$, $\log_2 |\tau'(x)|$ is a piecewise continuous function. Hence it follows from Lemma 2 that $f_k \to f$ weakly implies $\lim_{k \to \infty} \lambda_k = \lambda$. ∎

We now review the foregoing steps in the form of an algorithm.

Algorithm

Step 1. Let $\mathbf{I} = \{I_i\}_{i=0}^{N-1}$ be the partition of I into the intervals of smoothness of τ. Set

$$\mathbf{I}^{(k)} = \bigvee_{i=0}^{k} \tau^{-i}(\mathbf{I})$$

Let $Q^{(k)} = \{q_1^{(k)},..., q_{r_k}^{(k)}\}$ be the set of endpoints of the intervals that are elements of the partition $\mathbf{I}^{(k)}$.

Step 2. Form the piecewise linear Markov map τ_k on the partition $\mathbf{I}^{(k)}$ by choosing the images of the original points $\{a_j^-, a_j^+\}$ in such a way that τ_k has the magnitude of its slope greater than or equal to that of τ everywhere that it is defined.

Step 3. Let the matrix M_k denote the Frobenius–Perron operator of P_{τ_k} restricted to the space of step functions on $\mathbf{I}^{(k)}$. Compute the left eigenvector of M_k,[22] f_k, which we view as a step function on $\mathbf{I}^{(k)}$.

Step 4. Compute

$$\lambda_k = \sum_{\{i: I_i^{(k)} \in \mathbf{I}^{(k)}\}} f_k|_{I_i^{(k)}} m(I_i^{(k)}) \log_2 |\tau'_k|_{I_i^{(k)}}| \tag{2}$$

The algorithm was programmed in FORTRAN on an IBM PC. The partition points of $Q^{(n)}$ are found using the points in $Q^{(n-1)}$ as y values and then doing a binary search to find the corresponding x values. Each such x value becomes a new partition point unless it is within 0.001 of an existing partition point. During the nth (backward) iteration, it is only necessary to consider the partition points created during the $(n-1)$th iteration.

For $Q^{(k)}$, each interval of the associated partition $\mathbf{I}^{(k)}$ corresponds to one row of the matrix M_k. Recall

$$\begin{aligned} m_{ij} &= 1/|\tau'_k|_{I_i^{(k)}}| && \text{if} \quad \tau_k(I_i^{(k)}) \supseteq I_j^{(k)} \\ &= 0 && \text{otherwise} \end{aligned}$$

In computing

$$\tau_k \big|_{I_i^{(k)}}$$

we choose the closest partition point of $Q^{(k)}$ that produces a slope for τ_k larger in magnitude than that of

$$\tau \big|_{I_i^{(k)}}$$

Gaussian elimination is used to find the left eigenvector of M_k, which is then normalized by the requirement that

$$\sum_{\{i: I_i^{(k)} \in \mathbf{I}^{(k)}\}} f_k \big|_{I_i^{(k)}} m(I_i^{(k)}) = 1$$

Finally, the summation in (2) yields the estimate of the Liapunov exponent.

Remark. This algorithm can be applied to maps of the entire real line by using the results of Ref. 38.

Example 1. Consider the map $\tau: I \to I$ shown in Fig. 2. The map τ is a piecewise linear Markov map, whose Frobenius–Perron operator has the matrix representation

$$M_\tau = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

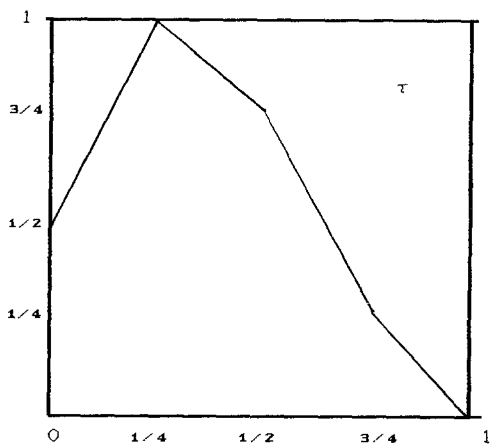

Fig. 2. A piecewise linear Markov Map.

Solving for the normalized left eigenvector, we get

$$f(x)=\begin{bmatrix} \frac{2}{7}, & 0\leqslant x<\frac{1}{4} \\ \frac{1}{7}, & \frac{1}{4}\leqslant x<\frac{1}{2} \\ \frac{2}{7}, & \frac{1}{2}\leqslant x<\frac{3}{4} \\ \frac{2}{7}, & \frac{3}{4}\leqslant x\leqslant 1 \end{bmatrix}$$

from which it follows that $\lambda=\frac{4}{7}$.

Since τ has $|\text{slope}|=1$ on two intervals, it is not an expanding map. But the foregoing analysis is valid since τ^2 is expanding. Applying the algorithm to τ, we obtained $\lambda=0.57143$ for every partition used.

Example 2. Consider the family of expanding maps

$$\tau_a(x)=\begin{cases} ax^2+(2-a/2)x, & 0\leqslant x\leqslant\frac{1}{2} \\ a(1-x)^2+(2-a/2)(1-x), & \frac{1}{2}\leqslant x\leqslant 1 \end{cases}$$

where $-2<a<2$, as shown in Fig. 3. Figure 4 shows the computed Liapunov exponent as a function of a.

Example 3. Consider the discontinuous, piecewise monotonic, expanding map $\tau\colon I\to I$ defined by

$$\tau(x)=\begin{cases} 0.2+0.6(x^2+\frac{3}{2}x), & 0\leqslant x<\frac{1}{2} \\ 0.1+0.8[(1-x)^2+\frac{3}{2}(1-x)], & \frac{1}{2}\leqslant x\leqslant 1 \end{cases}$$

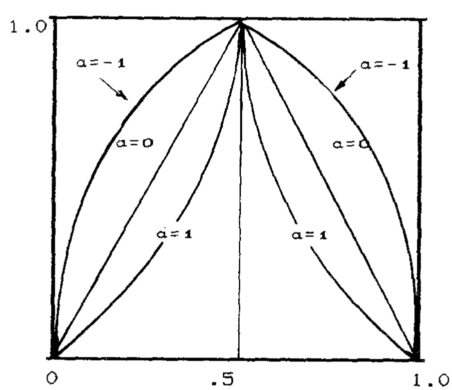

Fig. 3. A family of expanding maps.

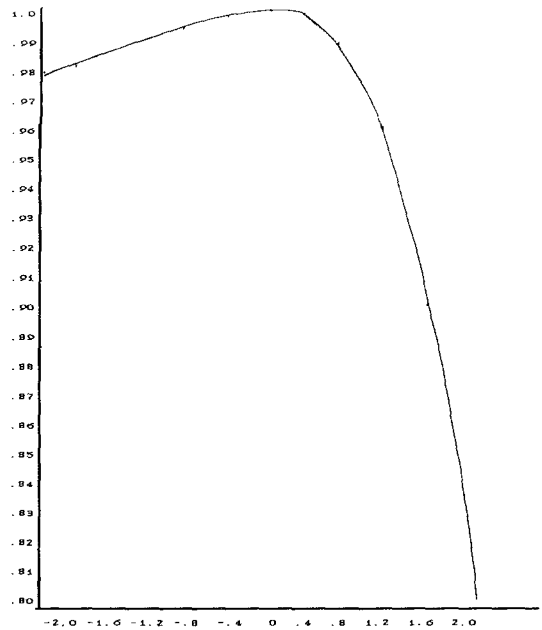

Fig. 4. Liapunov exponent as a function of a.

Table I displays the results obtained using the algorithm. The first column denotes the number of the backward iteration

$$\mathbf{I}^{(n)} = \bigvee_{i=0}^{n} \tau^{-i}(\mathbf{I})$$

The second column denotes the number of points in the nth-level partition and the third column shows the Liapunov exponent corresponding to the nth-level partition.

Table I

Number of iterations	Number of partition points	Estimate of Liapunov exponent
0	2	1.00
1	4	0.98904
2	7	0.89681
3	12	0.83514
4	31	0.77115
5	55	0.71509
6	97	0.71109
7	12	0.70346
8	144	0.70317

4. NONEXPANDING MAPS

Unlike the situation for piecewise, monotonic, expanding maps, there is no general theorem for nonexpanding maps that guarantees the existence of an absolutely continuous invariant measure. In Refs. 23–26 some special results are proved. The essential condition that appears in Refs. 23, 24, and 26 is that the Schwarzian be negative. In Ref. 26 this condition is weakened slightly and necessary and sufficient conditions provided for the existence of an absolutely continuous invariant measure. Indeed, for certain families of maps $\{\tau_\beta\}$, τ_β has a set of parameter values of positive Lebesgue measure.[26]

Although the negative Schwarzian condition may seem restrictive, it is satisfied by many dynamical systems of physical importance, for example, the famous logistic transformation $\tau_\beta(x) = 4\beta x(1-x)$, which models population dynamics, and the transformation $\tau(x) = rxe^{-bx}$, which is an accurate model for the Poincaré sections of the Belousov–Zhabotinski reaction in a well-stirred flow reactor.[1,4,36,37]

In this section we assume that τ is a unimodal map that has a negative Schwarzian and admits an absolutely continuous measure μ. We show that in this situation the algorithm of Section 3 applies, i.e., the densities of the approximating Markov maps approach the density of μ weakly. Hence this permits the approximation of the Liapunov exponent as in Section 3.

The reason for assuming τ to have a negative *Schwarzian* is that it guarantees that the inverse images of the turning point are dense in I. If this were known or could be proved, then the negative Schwarzian condition is unnecessary.

Definition 3. Let $\tau: I \to I$ be in C^3, i.e., it has three continuous derivatives. Then the Schwarzian derivative of τ is

$$S(\tau) = \frac{\tau'''(x)}{\tau'(x)} - \frac{3}{2}\left(\frac{\tau''(x)}{\tau'(x)}\right)^2$$

Now let

$$\mathbf{T} = \{\tau : \tau \in C^3, \tau \text{ is unimodal, and } S(\tau) < 0\}$$

Lemma 1. Let $\tau \in \mathbf{T}$ and assume it has an absolutely continuous invariant measure. Then $\bigcup_{k=0}^{\infty} \tau^{-k}(c)$ is dense in I, where c is the initial point of τ.

Proof. In Ref. 32 it is shown that if τ has a stable periodic orbit, then the set of points that is not attracted to the stable periodic orbit has Lebesgue measure 0. Hence, in this case, τ cannot have an absolutely continuous invariant measure, since the support of this measure has positive Lebesgue measure. Thus, τ cannot have a stable periodic orbit and it follows from Ref. 32 or Lemma 3 of Ref. 26 that $\bigcup_{k=0}^{\infty} \tau^{-k}(c)$ is dense in I. ▮

We shall require the following definition and two preliminary results.

Definition 4. Let g be any function from I into $(-\infty, \infty)$, and let δ and ε be positive numbers. We denote by $\partial_{\delta,\varepsilon}(g)$ the set of those points $x \in I$ for which the distance between $g(x')$ and $g(x'')$ exceeds ε for some pair of points x', x'' in the open interval $(x-\delta, x+\delta)$.

A more general version of the following theorem is proved in Ref. 29.

Theorem 2. Let $\{g_n\}_{n \geqslant 1}$ be a sequence of bounded, real-valued, and measurable functions defined on S and let α be a real number. Then a necessary and sufficient condition that $\int_I g_n(x) f_n(x)\, m(dx) \to \alpha$ for every sequence $\{f_n\}$ converging weakly to f is that

(a) $\{g_n\}_{n \geqslant 1}$ is uniformly bounded

(b) $\int_I g_n(x) f(x)\, m(dx) \to \alpha$

(c) $\forall \varepsilon > 0$, $\lim_{\delta \to 0} \limsup_{n \to \infty} \int_{\partial_{\delta,\varepsilon}(g_n)} f(x)\, m(dx) = 0$

It can be shown that (c) holds iff

(c′) $\forall \varepsilon > 0$, for every sequence $\{\delta_k\}$ of positive numbers converging to 0, and for every subsequence $\{g_{n_k}\}$,

$$\int_{\bigcap_{k=1}^{\infty} \partial_{\delta_k,\varepsilon}(g_{n_k})} f(x)\, m(dx) = 0$$

Lemma 2. Let g be a bounded, piecewise continuous function on $[0, 1]$ whose set of discontinuity points D has Lebesgue measure 0. Let $\{g_n\}$ be a uniformly bounded sequence of piecewise continuous functions which approaches g uniformly. Then, if $f_n \to f$ weakly as $n \to \infty$,

$$\int_I g_n(x) f_n(x)\, m(dx) \to \int_I g(x) f(x)\, m(dx)$$

as $n \to \infty$.

Proof. Since $g_n \to g$ uniformly, we have

$$\int_I g_n(x) f(x)\, m(dx) \to \int_I g(x) f(x)\, m(dx)$$

It remains to prove (c′). Let $\varepsilon > 0$. Then for any sequence $\{\delta_k\}$ of positive numbers converging to 0 and every subsequence $\{g_{n_k}\}$, $\bigcap_{k=1}^{\infty} \partial_{\delta_k,\varepsilon}(g_{n_k}) \subset D$. Since $m(D) = 0$, (c′) is valid and Theorem 2 can be invoked. ▮

Lemma 3. Let $\{\tau_n\}$ be a sequence of nonsingular transformations from $I \to I$ that approach τ uniformly. Let $f \in \mathbf{L}_1$. Then

$$\int_I h(x)(P_{\tau_n} f)(x)\, m(dx) \to \int_I h(x)(P_\tau f)(x)\, m(dx)$$

as $n \to \infty$ for any $h \in C^1$, the space of functions on I that have continuous first derivative.

Proof. From the definition of the Frobenius–Perron operator, we have

$$\begin{aligned}\int_I h(x)[P_{\tau_n} f(x) - P_\tau f(x)]\, m(dx) \\ = \int_I h(x) \left[\frac{d}{dx} \int_{\tau_n^{-1}[0,x]} f(y)\, m(dy) \right. \\ \left. - \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(y)\, m(dy) \right] m(dx)\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^1 h(x) \left[\frac{d}{dx} \int_{\tau^{-1}[0,x]} f(y)\, m(dy) \right] m(dx) \\ = g(1) \int_0^1 f(x)\, m(dy) \\ - \int_0^1 \int_{\tau^{-1}[0,x]} f(y)\, m(dy)\, g'(x)\, m(dx)\end{aligned}$$

Thus

$$\begin{aligned}\int_I h(x)[P_{\tau_n} f(x) - P_\tau f(x)]\, m(dx) \\ = \int_I \left[\int_{\tau^{-1}[0,x]} f(y)\, m(dy) \right. \\ \left. - \int_{\tau_n^{-1}[0,x]} f(y)\, m(dy) \right] h'(x)\, m(dx)\end{aligned}$$

and

$$\begin{aligned}\left| \int_I h(x)[P_{\tau_n} f(x) - P_\tau f(x)]\, m(dx) \right| \\ \leqslant \int_I \int_{(\tau^{-1}[0,x]) \Delta (\tau_n^{-1}[0,x])} |f(y)|\, m(dy)\, h'(x)\, m(dx)\end{aligned}$$

where Δ denotes the symmetric difference. Since $\tau_n \to \tau$ uniformly as $n \to \infty$,

$$m\{(\tau^{-1}[0, x])\,\Delta(\tau_n^{-1}[0, x])\} \to 0 \qquad \text{as} \quad n \to \infty$$

Since $h'(x)$ is continuous on I, it is bounded. This completes the proof. ∎

We can now state the main result of this section.

Theorem 3. Let $\tau: I \to I$ be a unimodal map such that $\bigcup_{k=0}^{\infty} \tau^{-k}(c)$ is dense in I, where c is the critical point, and such that τ admits an absolutely continuous invariant measure μ with probability density function f^*. Let $\{\tau_n\}$ be a sequence of piecewise linear Markov maps such that $\tau_n \to \tau$ uniformly as $n \to \infty$. Let f_n be the probability density function of the measure invariant under τ_n. Assume $f_n \to f$ weakly as $n \to \infty$. Then $f = f^*$.

Proof. Let P_n be the Frobenius–Perron operator of τ_n. Then $P_n f_n = f_n$. Now, for any $h \in C^1$,

$$\begin{aligned}
&\left| \int_I h(x)[f(x) - P_\tau f(x)]\, m(dx) \right| \\
&\quad \leqslant \left| \int_I h(x)[f(x) - f_n(x)]\, m(dx) \right| \\
&\qquad + \left| \int_I h(x)[f_n(x) - P_n f_n(x)]\, m(dx) \right| \\
&\qquad + \left| \int_I h(x)[P_n f_n(x) - P_n f(x)]\, m(dx) \right| \\
&\qquad + \left| \int_I h(x)[P_n f(x) - P_\tau f(x)]\, m(dx) \right|
\end{aligned}$$

The first term approaches 0, since $f_n \to f$ weakly. Since $P_n f_n = f_n$, the second term is identically 0. The fourth term approaches 0 by virtue of Lemma 3. Consider now the third term,

$$\begin{aligned}
&\int_0^1 h(x) \frac{d}{dx} \left\{ \int_{\tau_n^{-1}[0,x]} [f_n(y) - f(y)]\, m(dy) \right\} m(dx) \\
&\quad = \int_0^1 \left\{ \int_{\tau_n^{-1}[0,x]} [f_n(y) - f(y)]\, m(dy) \right\} h'(x)\, m(dx)
\end{aligned}$$

Fix $x \in [0, 1]$ and consider

$$A_n(x) \equiv \int_{\tau_n^{-1}[0,1]} [f_n(y) - f(y)]\, m(dy)$$

$$= \int_I \chi_{\tau_n^{-1}[0,1]}(y) f_n(y)\, m(dy) - \int_I \chi_{\tau_n^{-1}[0,1]}(y) f(y)\, m(dy)$$

Now $\chi_{\tau_n^{-1}[0,1]}(y)$ is a piecewise continuous step function, which approaches $\chi_{\tau^{-1}[0,1]}$ uniformly as $n \to \infty$. Clearly

$$\int_I \chi_{\tau_n^{-1}[0,1]}(y) f(y)\, m(dy) \to \int_I \chi_{\tau_n^{-1}[0,1]}(y) f(y)\, m(dy)$$

as $n \to \infty$. Thus, it follows from Lemma 2 that $A_n(x) \to 0$ as $n \to \infty$. Note that $|A_n(x)| \leqslant 2$. Since $h \in C^1$, $|h'(x)| \leqslant L < \infty$. Hence, the dominated convergence theorem implies that

$$\int_0^1 A_n()\, h'(x)\, m(dx) \to 0$$

as $n \to \infty$. We have therefore established that, for any $h \in C^1$,

$$\int_I h(x)[f(x) - P_\tau(x)]\, m(dx) = 0$$

This means $P_\tau f(x) = f(x)$ m-a.e. But f^* is the unique fixed point of P_r. Thus, $f = f^*$ m-a.e., and $f \to f^*$. ∎

Corollary 2. Let λ_k be the Liapunov exponent of τ_k, as in (2). Then

$$\lambda_k \to \lambda = \int_0^1 f(x) \log_2 |\tau'(x)|\, m(dx)$$

which is the Liapunov exponent of τ, as $k \to \infty$.

Proof. Weak convergence of f_k to f as $k \to \infty$. ∎

Thus, the algorithm of Section 3 is valid for τ satsfying the conditions of Theorem 3, in particular if $\tau \in \mathrm{T}$ has an absolutely continuous invariant measure.

Remarks. 1. The weakness in Theorem 3 is the critical assumption that $f_n \to f$ weakly as $n \to \infty$. In the case when τ is expanding, this follows from the main result of Ref. 9. In the nonexpanding case this has recently been proved in Ref. 35 for nonexpanding maps which are conjugate to piecewise expanding maps via an absolutely continuous homeomorphism.

Table II

Number of iterations	Number of partition points	Estimate of Liapunov exponent
0	2	1.00
1	4	0.87243
2	8	0.92392
3	16	0.95968
4	32	0.97934
5	64	0.98957
6	128	0.99475

2. All that is needed for Theorem 3 to work is that $\tau_n \to \tau$ uniformly as $n \to \infty$ and that $\{f_n\}$ converges weakly to some $f \in \mathrm{L}_1$ as $n \to \infty$.

3. Since τ is unimodal, it can have only one absolutely continuous invariant measure.[23,24]

Example 4. Consider the logistic map $\tau(x) = 4x(1-x)$. Since τ is differentiably conjugate to the triangle map with slope $+2$ and -2, the Liapunov exponent is 1. We applied the algorithm of Section 3 to τ and obtained the results shown in Table II.

5. THE PRESENCE OF NOISE

For the expanding maps of Section 3, it is shown in Refs. 11 and 34 that the probability density function of the measure invariant under τ is stable under small, random perturbations. Thus, the algorithm of Section 3 will yield stable results in the presence of noise. For nonexpanding maps such as those considered in Section 4, it is shown in Ref. 13 that measures of the noisy system converge to the absolutely continuous measure, which exists in virtue of the results in Ref. 23. Problems related to estimating Liapunov exponents in the presence of the noise are discussed in Section 7.2 of Ref. 2.

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REFERENCES

1. A. Wolf and J. Swift, in *Statistical Physics and Chaos in Fusion Plasmas*, C. W. Horton, Jr., and L. E. Reichl, eds. (Wiley, 1984).
2. A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, *Physica* **16D**:285–317 (1985).
3. J.-C. Roux, R. H. Simoyi, and H. L. Swinney, *Physica* **8D**:257–266 (1983).
4. A. Wolf, in *Chaos*, A. V. Holden, zs. (Manchester University Press, 1986).
5. A. Wolf and J. A. Vastano, in *Dimension and Entropies in Chaotic Systems*, G. Mayer-Kress, ed. (Springer-Verlag, 1987).
6. J. Wright, *Phys. Rev. A* **29**(5):2924–2927 (1984).
7. L.-S. Young, *IEEE Trans. Circuits Systems* **CAS-30**(8):599–607 (1983).
8. F. Ledrappier, *Ergodic. Theory Dynam. Syst.* **1**:77–93 (1981).
9. A. A. Kosyakin and E. A. Sandler, *Izv. VUZ Matemat.* (3) [118], 32–40 (1972). (English translation available from the British Library, Translation Service.)
10. M. Sano and Y. Sawada, *Phys. Rev. Lett.* **55**(10):1082–1085 (1985).
11. A. Boyarsky, *J. Math. Anal. Appl.* **76**:483–497 (1980).
12. G. Choquet, *Lectures on Analysis*, Vol. I (Benjamin, New York, 1969).
13. A. Katok and Y. Kifer, *Tour. Anal. Math.* **47**:193–237 (1986).
14. A. Boyarsky, Singular perturbations of piecewise monotonic maps of the interval, preprint.
15. H. Proppe, W. Byers, and A. Boyarsky, *Isr. J. Math.* **44**(4):277–288 (1983).
16. V. I. Oseledec, *Trans. Moscow Math. Soc.* **19**:197–231 (1968).
17. A. M. Liapunov, *Ann. Math. Study* **17** (1947).
18. G. Bennetin, L. Galgani, and J. M. Strelcyn, *Meccanica* **15**:9–20 (1980).
19. H. A. Lauwener, in *Chaos*, A. V. Holden, ed. (Princeton University Press, 1986).
20. M. Denker, C. Gillenberger, and K. Sigmund, *Ergodic Theory on Compact Spaces* (Springer-Verlag, New York, 1976).
21. A. Lasota and J. A. Yorke, *Trans. Am. Math. Soc.* **186**:481–488 (1973).
22. A. Boyarsky and M. Scarowsky, *Trans. Am. Math. Soc.* **255**:243–262 (1979).
23. M. Misiurewicz, *Inst. Hautes Etudes Sci. Publ. Math.* **53** (1981).
24. M. Misiurewicz, *Inst. Hautes Etudes Sci. Publ. Math.* **51** (1979).
25. R. Bowen, *Commun. Math. Phys.* **69**:1–17 (1979).
26. S. D. Johnson, Continuous measures and strange attractors in one dimension, Ph. D. Dissertation, Stanford University (1985).
27. N. Friedman and A. Boyarsky, *Lin. Alg. Appl.* **38**:141–147 (1981).
28. G. Choquet, *Lectures on Analysis*, Vol. I (Benjamin, New York, 1969).
29. F. Topsoe, *Ann. Math. Stat.* **38**:1661–1665 (1967).
30. B. L. S. Prakasa Rao, *Non Parametric Functional Estimation* (Academic Press, 1983).
31. L. Arnold and V. Wihstutz, Liapunov exponents: A survey, Report No. 141, University of Bremen (1985).
32. J. Guckenheimer, *Commun. Math. Phys.* **70**:133–160 (1979).
33. L. Arnold, Stochastic systems: Qualitative theory and Liapunov exponents, preprint.
34. P. Góra, *Coll. Math.* **XLIL**:73–85 (1984).
35. P. Góra, A. Boyarsky, H. Proppe, Constructive approximations to densities invariant under non-expanding transformations, *Jour. Stat. Physics*, in press.